Vibration-induced interfacial instabilities in viscoelastic fluids

Satish Kumar

*Department of Chemical Engineering and Materials Science, University of Minnesota, 151 Amundson Hall, 421 Washington Avenue SE,
Minneapolis, Minnesota 55455*

Vertically vibrated interfaces between viscoelastic fluids may arise in contexts as diverse as ultrasonic emulsification, microgravity materials processing, and geophysics. If the vibration amplitude is large enough at a given frequency, the interface can become unstable and give rise to standing waves. The present work provides a linear analysis of this phenomenon for the cases where either or both of the fluids are viscoelastic. The fluids are assumed to be laterally unbound, and Floquet theory is used to develop a recursion relation between the temporal modes of the interfacial deformation. Conversion of this relation into a matrix eigenvalue problem allows determination of the critical vibration amplitude needed to excite the standing waves and the corresponding critical wave number. Using a single-mode Maxwell model to describe the viscoelasticity and considering infinite fluid depths, we present calculations for three cases: bottom fluid viscoelastic/top fluid Newtonian, bottom fluid Newtonian/top fluid viscoelastic, and both fluids viscoelastic. When only one of the fluids is viscoelastic, the interfacial waves can respond harmonically to the forcing. The waves may also be excited more easily than in the case where both fluids are Newtonian. When both of the fluids are viscoelastic, it appears possible to excite Stoneley-like waves at the interface.

I. INTRODUCTION

When a flat free surface is subject to vertical vibration, it may become unstable to standing waves. This was first reported by Faraday [1], and over a century later the mechanism was explained by Benjamin and Ursell [2]. The mechanism is that of parametric resonance, and is exemplified by a pendulum whose pivot point oscillates vertically [3]. This problem has been thoroughly examined for inviscid [2,4], Newtonian [5–7], and viscoelastic [8–11] fluids. Linear stability analysis accurately predicts the critical vibration amplitude needed to excite the surface waves, as well as the corresponding critical wave number. A similar instability and mechanism operate when the interface between two immiscible fluids is subject to vertical vibration. While the case where both fluids are Newtonian has been well studied, the cases where either or both fluids are viscoelastic have not been examined. The latter cases are of potential importance to several practical applications and are the topic of the present study.

The excitation of interfacial waves by vertical vibration is relevant to emulsification, microgravity materials processing, and geophysics. If the wave amplitude becomes sufficiently large, droplets may pinch off and produce an emulsion. Ultrasound can be used as a vibration source, and doing so for Newtonian fluids results in stable dispersions of submicron particles having a relatively uniform size distribution [12]. Applying this technique to viscoelastic fluids may lead to the creation of novel polymeric nanocomposite materials. In microgravity environments, viscoelastic fluids may appear as polymer solutions and melts, colloidal suspensions, or biological media (e.g., cells suspended in solution). If these fluids are in contact with another fluid and undergo vertical vibration, the resulting interfacial waves may be detrimental to the process of which the fluids are a part. On geophysical scales, liquid-like sediment could possess viscoelastic rheo-

logical properties and be in contact with water or other sediment [13]. Having a way to compute the critical amplitude and wave number of vibration-induced interfacial waves might prove useful in predicting some of the effects of earthquakes.

Vibration-induced waves at the interface between two Newtonian fluids have been analyzed by a number of different investigators. Motivated by the observation that vertical vibration can suppress the Rayleigh-Taylor instability, Troyon and Gruber applied Fourier and Laplace transforms to the linearized governing equations [14]. Their predictions of the critical vibration amplitude and wave number agreed favorably with the experiments of Hoffmann and Wolf [15]. The same problem was also considered by Jacqmin and Duval, who applied Floquet theory [16]. In both of the above studies, the fluids were assumed to be of infinite depth. Finite-depth effects were accounted for by Hasegawa, Nagashima, and Takashima [17], as well as by Kumar and Tuckerman [18]. Again, the theoretical predictions were found to be in good agreement with experimental measurements [17,19].

In this work, we extend the above studies to cases where one or both of the fluids are viscoelastic. We apply Floquet theory to the linearized governing equations and demonstrate how one can calculate the critical vibration amplitude needed to excite standing waves at the interface, and the corresponding critical wave number. In Sec. II, we set up the problem and explain the calculation method. Results are presented in Sec. III, where we consider three cases: bottom fluid viscoelastic/top fluid Newtonian, bottom fluid Newtonian/top fluid viscoelastic, and both fluids viscoelastic. Finally, a discussion and conclusions are given in Sec. IV.

II. PROBLEM SETUP

The base state for the problem is shown in Fig. 1. Two immiscible fluids whose interface is flat undergo a vertical
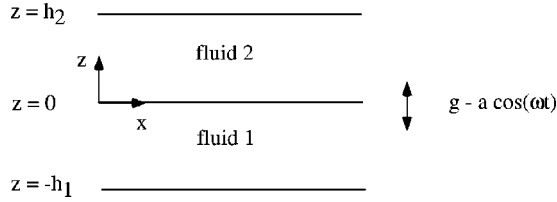
FIG. 1. Interface between two viscoelastic fluids under vertical vibration. The reference frame moves with the vibration so that the interface is located at $z=0$ in the base state. The fluids are bounded by solid walls at $z=-h_1$ and $z=h_2$. The vibration occurs at a frequency ω and with an amplitude a. The mean gravity g acts in the negative-z direction, and the top fluid is taken to have a smaller density than the bottom fluid.

oscillation of amplitude a and frequency ω, with a mean gravity g acting in the negative-z direction. We denote the bottom fluid as "fluid 1" and the top fluid as "fluid 2." Their densities are given by ρ_1 and ρ_2, and we assume that $\rho_2 < \rho_1$ so that the interface is stable in the absence of vibration. A tension σ acts at the interface. Adopting a reference frame that moves with the oscillation, we take the flat interface to be located at $z=0$ and the fluids to be bounded by solid walls at $z=h_2$ and $z=-h_1$. The fluids are assumed to be laterally unbound, and this will be a good assumption when the critical wavelength is much less than the container width.

In linear stability analysis, we are interested in small departures from the base state, and in this problem these correspond to small deformation gradients since the base state is quiescent. As a consequence, the extra-stress tensors for the fluids can be described by the theory of linear viscoelasticity [20],

$$\tau_j = \eta_{s,j}[\boldsymbol{\nabla}\mathbf{u}_j + (\boldsymbol{\nabla}\mathbf{u}_j)^T] + \int_{-\infty}^{t} G_j(t-t')[\boldsymbol{\nabla}\mathbf{u}_j + (\boldsymbol{\nabla}\mathbf{u}_j)^T]dt',$$

$$(1)$$

where the $\mathbf{u}_j$ are the velocity fields, the G_j are the relaxation moduli, and the $\eta_{s,j}$ are the solvent viscosities for each fluid ($j=1$ or 2).

Because the reference frame is moving, the equations of motion for each fluid will have time-dependent body forces given by $-\rho_j B(t)\mathbf{e}_z$, where $B(t)=g-a\cos(\omega t)$ and e_z is the unit vector in the z direction. Recognizing that in the base state the pressure fields are time dependent, and that the velocities of the fluids are zero, leads us to the following linearized momentum and continuity equations:

$$\rho_j \partial_t \mathbf{u}_j = -\boldsymbol{\nabla}p_j + \eta_{s,j}\nabla^2 u_j + \int_{-\infty}^{t} G_j(t-t')\nabla^2 \mathbf{u}_j dt', \quad (2)$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}_j = 0. \quad (3)$$

All variables now refer to disturbance quantities. By applying the operator $\mathbf{e}_x\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times$ to Eq. (2) and using Eq. (3), we can obtain equations for the z components of the velocities w_j,

$$(\partial_t - \nu_{s,j}\nabla^2)\nabla^2 w_j - \frac{1}{\rho_j}\int_{-\infty}^{t} G_j(t-t')\nabla^2(\nabla^2 w_j)dt' = 0,$$

$$(4)$$

where $\nu_{s,j} = \eta_{s,j}/\rho_j$.

Equation (4) is subject to boundary conditions at the solid walls (no-slip and no-penetration) and at the interface (continuity of velocity, and balance of tangential and normal force components). Because these boundary conditions have forms very similar to those given in [9] and [18], we do not reproduce them here. The interface location, $\zeta=\zeta(x,y,t)$, is described by the kinematic condition [18].

The normal-mode decomposition is performed by replacing $w_j(x,y,z,t)$ with $w_j(z,t)\exp(i\mathbf{k}\cdot\mathbf{x})$ and $\zeta(x,y,t)$ with $\zeta(t)\exp(i\mathbf{k}\cdot\mathbf{x})$. Here, $\mathbf{k}$ is a real-valued wave vector (whose magnitude is denoted by k) and $\mathbf{x}$ is a coordinate vector, both of which are in the x-y plane. Since the system under study is periodically forced, Floquet theory can be applied to describe its time dependence. Fourier series are used to enforce the $2\pi/\omega$ periodicity of ζ and the w_j,

$$\zeta(t) = e^{(s+i\alpha\omega)t}\sum_{n=-\infty}^{\infty}\zeta_n e^{in\omega t}, \quad (5)$$

$$w_j(z,t) = e^{(s+i\alpha\omega)t}\sum_{n=-\infty}^{\infty}w_{j,n}(z)e^{in\omega t}, \quad (6)$$

where s is a real-valued growth rate and the value of α determines whether the surface responds subharmonically ($\alpha=1/2$) or harmonically ($\alpha=0$) to the forcing. Previous work on vibration-induced instabilities suggests that these are the only types of responses which need to be considered [5,11,16,18].

Application of Eqs. (5) and (6) to the governing equations and boundary conditions, along with a number of further manipulations like those in [9] and [18], leads to a recursion relation for the ζ_n,

$$A_n\zeta_n = a(\zeta_{n+1} + \zeta_{n-1}), \quad (7)$$

where A_n depend on k, the complex growth rate $s+i(\alpha+n)\omega$, the physical properties of the fluids, the fluid depths, and the forcing frequency. The expressions for the A_n are the same as those used in [18] except that they now involve a frequency-dependent wave number and viscosity,

$$q_{j,n}^2 = k^2 + \frac{s+i(\alpha+n)\omega}{\nu_{j,n}(\omega)}, \quad (8)$$

with

$$\nu_{j,n}(\omega) = \frac{\eta_{s,j}}{\rho_j} + \frac{1}{\rho_j}\int_0^{\infty} G_j(\tau)\exp\{-[s+i(\alpha+n)\omega]t\}d\tau. \quad (9)$$

A matrix eigenvalue problem can be formed from the recursion relation (7) by truncating it at a finite value, say $n=N$ [18]. The eigenproblem has the form
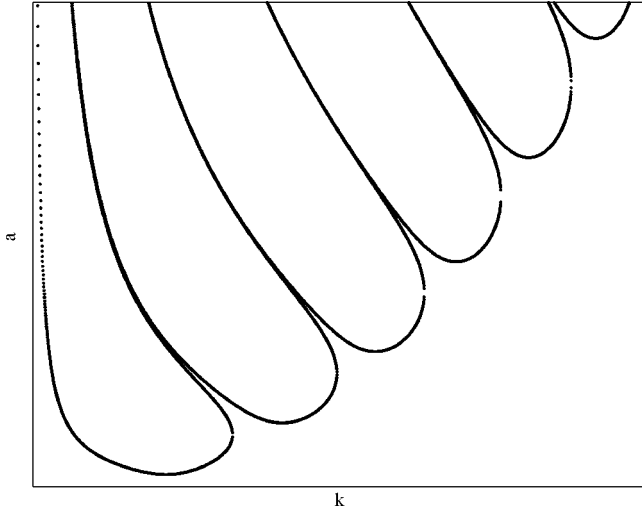
FIG. 2. Tonguelike neutral stability curves in the k-a plane. The tongues alternately correspond to subharmonic and harmonic responses of the interfacial waves to the forcing, with the tongue closest to the a-axis corresponding to a subharmonic response.

$$\mathbf{A}^{-1}\mathbf{B}\zeta = \frac{1}{a}\zeta. \tag{10}$$

The matrices $\mathbf{A}$ and $\mathbf{B}$ are real and of dimension $2(N+1) \times 2(N+1)$, and the vector ζ is of dimension $2(N+1) \times 1$. At each value of k and ω, one can compute the eigenvalues $1/a$ using standard algorithms. Doing this over a range of k values at fixed ω and plotting the results in the k-a plane produces a sequence of tonguelike curves (Fig. 2). One set of curves can be generated for $\alpha = 1/2$ (subharmonic response) and another for $\alpha = 0$ (harmonic response). The neutral stability curves are those for which $s = 0$, and the critical vibration amplitude and corresponding wave number are given by the tongue tip closest to the k-axis.

III. RESULTS

As we are interested in uncovering the most basic effects of viscoelasticity, we take both fluids to be of infinite depth in our calculations. In addition, we adopt the single-mode Maxwell model to describe the relaxation moduli,

$$G_j(t-t') = \frac{\eta_{p,j}}{\lambda_j}\exp[-(t-t')/\lambda_j], \tag{11}$$

where the λ_j are the relaxation times for each fluid. If we imagine that the viscoelasticity arises due to polymers dissolved in a Newtonian solvent, then the $\eta_{p,j}$ represent the polymer contributions to the zero-shear viscosity for each fluid. The ratios $\eta_{p,j}/\lambda_j$ are simply the shear moduli.

With the single-mode Maxwell model, the complex-valued frequency-dependent viscosity (9) becomes

$$\nu_{j,n}(\omega) = \nu_{s,j} + \frac{\nu_{p,j}}{1+i(\alpha+n)De_j}, \tag{12}$$

where $\nu_{p,j} = \eta_{p,j}/\rho_j$ and $De_j = \lambda_j\omega$. The De_j are the Deborah numbers of each fluid, and give the ratio of the fluid relaxation time to a characteristic time of the standing wave oscillations. If $De_j \ll 1$ in a given fluid, then the non-Newtonian stresses relax relatively quickly and the fluid behaves in a Newtonian manner with $\nu_{j,n}(\omega) = \nu_{s,j} + \nu_{p,j}$. If $De_j \gg 1$, then the non-Newtonian stresses relax relatively slowly and Newtonian behavior also occurs with $\nu_{j,n}(\omega) = \nu_{s,j}$.

In our work, we have obtained results for three cases: bottom fluid viscoelastic/top fluid Newtonian, bottom fluid Newtonian/top fluid viscoelastic, and both fluids viscoelastic. The code for computing the critical vibration amplitude and wave number was validated by reproducing known results for Newtonian systems [18]. We used ten Fourier modes ($N = 10$) in solving the matrix eigenvalue problem, and verified that this was enough to give accurate solutions.

A. Bottom fluid viscoelastic, top fluid Newtonian

Before considering the instability at the interface between a viscoelastic fluid and a Newtonian fluid, it is worthwhile to review what happens when only the viscoelastic fluid is present (i.e., when the instability occurs at a free surface) [9]. To see a distinct non-Newtonian effect on instability behavior, elastic forces must be at least as strong as viscous forces, which implies that $\eta_p \approx \eta_s$. If this is the case, and if surface tension forces are sufficiently strong relative to elastic forces, the surface waves will respond subharmonically to the forcing. The critical wave number will be similar to that for a Newtonian fluid of the same zero-shear viscosity, but the critical amplitude will increase less rapidly with the vibration frequency. As η_p increases at fixed η_s and σ, elastic forces become stronger relative to viscous forces and surface-tension forces. In this case, it is possible for the surface waves to respond harmonically to the forcing provided that De is not too large or too small. The presence of the harmonic response also requires λ to be below a critical value, which means that the elastic modulus must be above a critical value.

The harmonic response of viscoelastic fluids can occur when the fluid is of infinite depth, and is notable because Newtonian fluids of infinite depth airways respond subharmonically. The harmonic response in viscoelastic fluids appears to arise because such fluids can behave like elastic solids and thus support Rayleigh-like surface waves [9]. For Newtonian fluids that are relatively viscous ($\nu \sim 1$ cm²/s), harmonic responses can occur in shallow layers (~ 1 mm) [5]. Here, the viscous dissipation at the bottom boundary increases more rapidly for longer wavelengths as the fluid depth decreases. As a result, the tip of the subharmonic tongue closest to the a-axis in the neutral stability curves (Fig. 2) gets pushed to a higher value of a than that of the neighboring harmonic tongue. In viscoelastic fluids of finite depth, this mechanism for producing a harmonic response operates in addition to the mechanism discussed above.

Since the infinite-depth harmonic response is the most distinguished feature of the free-surface instability for a viscoelastic fluid, our results for interfacial instability will focus
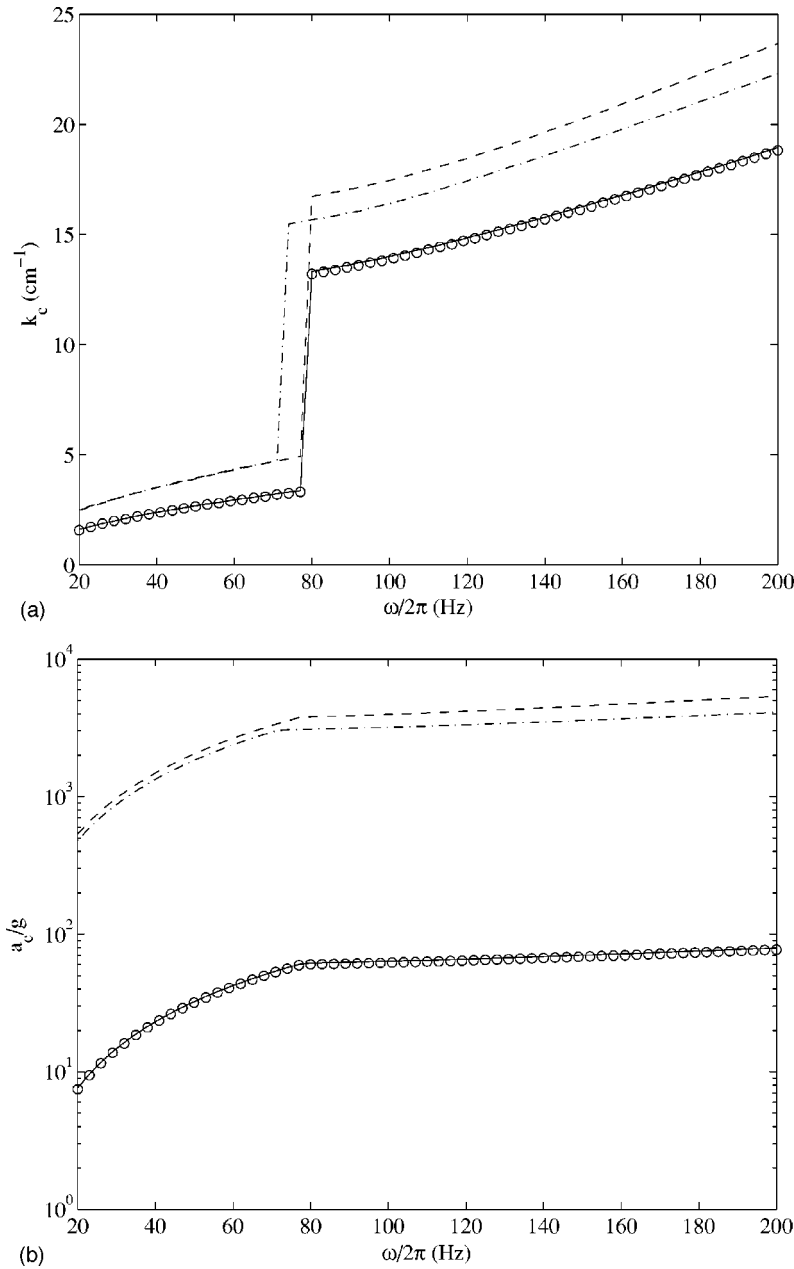
FIG. 3. The (a) critical wave number (k_c) and (b) critical vibration amplitude (a_c) vs the vibration frequency (ω) for the case where the bottom fluid is viscoelastic and the top fluid is Newtonian. Open circles $\rho_2/\rho_1=0.1$, $\nu_{s,2}/\nu_{s,1}=1$; Dash-dot line: $\rho_2/\rho_1=0.98$, $\nu_{s,2}/\nu_{s,1}=1$; Dashed line: $\rho_2/\rho_1=0.98$, $\nu_{s,2}/\nu_{s,1}=100$; Solid line: $\rho_2/\rho_1=0.1$, $\nu_{s,2}/\nu_{s,1}=100$.

on how the harmonic response is affected by a top fluid that is Newtonian. To facilitate comparison with previous studies, our calculations are done in dimensional units. We consider a viscoelastic fluid corresponding to a semidilute polymer solution, where $\rho_1=1$ g/cm^3, $\nu_{s,1}=0.01$ cm^2/s, $\nu_{p,1}=10$ cm^2/s, and $\lambda_1=1\times10^{-3}$ s. The interfacial tension σ is taken to be 10 dyne/cm. The viscosity ($\nu_{s,2}$) and density (ρ_2) of the top fluid are varied in the calculations ($\nu_{p,2}$ and λ_2 are zero.) We also consider vibration frequencies between 20 and 200 Hz, which correspond to mechanical agitation. For ultrasonic frequencies, $De_1\gg1$ and the harmonic response will not be present. The mean gravity g is taken to be that of earth, 981 cm^2/s.

Figure 3(a) shows plots of the critical wave number versus the vibration frequency for different values of the ratios ρ_2/ρ_1 and $\nu_{s,2}/\nu_{s,1}$. Figure 3(b) shows the corresponding plots of the critical vibration amplitude. For $\rho_2/\rho_1=0.1$ and

$\nu_{s,2}/\nu_{s,1}=1$, the response is similar to what would occur if a free surface were present. At around 80 Hz, there is a transition from a subharmonic to a harmonic response and the wave number jumps to a larger value. As ρ_2/ρ_1 increases to 0.98, the transition frequency decreases to around 74 Hz. The critical wave numbers and amplitudes also become larger. When $\nu_{s,2}/\nu_{s,1}$ is then increased to 100, the transition frequency increases back to around 80 Hz. The wave numbers (at least after the transition) become larger, as do the amplitudes. Finally, as ρ_2/ρ_1 drops to 0.1 the transition frequency remains at about the same value, but the critical wave numbers and amplitudes decrease. The plots for $\nu_{s,2}/\nu_{s,1}=1$ and $\nu_{s,2}/\nu_{s,1}=100$ at $\rho_2/\rho_1=0.1$ are approximately the same, due to the fact that the density of the top fluid is very small compared to that of the bottom.

From the above results, as well as from other runs we have performed, we can draw the following conclusions. At a

fixed solvent viscosity ratio, increasing the density of the top layer relative to that of the bottom decreases the transition frequency and increases the critical amplitudes. The higher amplitudes imply that the instability becomes harder to excite, and this is consistent with what we expect since density differences amplify the instability. The critical wave numbers, both before and after the subharmonic-to-harmonic transition, also tend to increase. At a fixed density ratio, increasing the solvent viscosity of the top layer relative to that of the bottom increases the transition frequency and critical amplitudes. Here, the higher amplitudes result because the increased viscous dissipation provided by the upper fluid must be overcome in order to excite the instability. The critical wave numbers before the transition increase and then decrease as $\nu_{s,2}/\nu_{s,1}$ increases, while those after the transition increase. The behavior of the critical amplitudes as $\nu_{s,2}/\nu_{s,1}$ or ρ_2/ρ_1 vary does not appear to be a non-Newtonian effect since similar behavior is seen in the case where both fluids are Newtonian.

B. Bottom fluid Newtonian, top fluid viscoelastic

When both fluids are Newtonian and of infinite depth, the interfacial waves always respond subharmonically to the forcing. In the preceding section, we have seen that it is possible to obtain a harmonic response when the bottom fluid is viscoelastic. We have also found that such a response is possible when the top fluid is viscoelastic. Consider the case where $\nu_{s,1}=0.01 \, \text{cm}^2/\text{s}$, $\rho_1=1 \, \text{g/cm}^3$, and $\sigma=10 \, \text{dyne/cm}$ ($\nu_{p,1}$ and λ_1 are zero). If we use the same range of vibration frequencies as before and take $\nu_{s,2}/\nu_{s,1}=1$, $\nu_{p,2}/\nu_{s,2}=1000$, $\rho_2/\rho_1=0.98$, and $\lambda=1\times10^{-3}$ s, a transition from a subharmonic to a harmonic response is observed around 74 Hz.

As the parameters for the viscoelastic fluid are similar to those used before, the same criteria apply for obtaining a harmonic response; elastic forces must be sufficiently stronger than surface tension and viscous forces, and De_2 must not be too large or small. In addition, ρ_2/ρ_1 must be sufficiently large; the harmonic response will disappear if ρ_2 is made too small compared to ρ_1. When the harmonic response does occur, we find that increasing ρ_2/ρ_1 at fixed $\nu_{s,2}/\nu_{s,1}$ tends to decrease the transition frequency and increase the critical amplitudes. Before and after the subharmonic-to-harmonic transition, the critical wave numbers decrease. We also find that as the Newtonian fluid becomes more viscous at fixed ρ_2/ρ_1 (i.e., $\nu_{s,2}/\nu_{s,1}$ decreases), the transition frequencies and critical amplitudes increase. Before the transition, the critical wave numbers increase and then decrease, while after the transition the critical wave numbers increase. We note that the effects of changing the solvent viscosity ratio are similar to what was reported in the preceding section.

We can also compare the effects of a viscoelastic top fluid to those of a Newtonian one having the same zero shear viscosity. Figures 4(a) and 4(b) show the critical wave numbers and amplitudes for such a calculation. For the viscoelastic fluid, the parameters are the same as before except that $\nu_{p,2}/\nu_{s,2}=100$. The response is always subharmonic in this

case since $\nu_{p,2}/\nu_{s,2}$ is relatively small. The figures indicate that although the wave numbers in both systems are similar, the critical amplitude increases less rapidly with the vibration frequency when there is a viscoelastic fluid. This is because the viscous resistance to the instability, which is proportional to the real part of Eq. (12) decreases as De_2 increases. The behavior seen in Fig. 4(a) and 4(b) is analogous to that observed for surface waves when there is only one fluid [9,10].

C. Both fluids viscoelastic

Waves on the surface of an elastic solid are known as Rayleigh waves, and exhibit a linear relationship between wave frequency and wave number. Viscoelastic fluids behave like elastic solids when elastic forces become very strong compared to surface tension and viscous forces, and when deformations occur on time scales shorter than the fluid relaxation time. Indeed, when only the bottom fluid is present, it is possible to excite Rayleigh-like surface waves through vertical vibration [9]. The evidence for this comes from the fact that the slope of the frequency-wave-number relation for the fluid closely matches that for Rayleigh waves on an elastic solid having the same shear modulus and density as the fluid.

When both a bottom and top fluid are present, it is natural to ask whether it possible for vertical vibration to excite Rayleigh-like waves at the interface. Interfacial waves between two elastic solids are called Stoneley waves, and also obey a linear frequency-wave-number relationship [21,22]. For Rayleigh waves on incompressible solids, the slope of this relationship is simply $0.955(\mu/\rho)^{1/2}$, where μ is the shear modulus and ρ is the density. For Stoneley waves, the slope depends upon the modulus and density ratios in a much more complicated fashion. The slope is $y(\mu_2/\rho_2)^{1/2}$ where μ_2 and ρ_2 are the shear modulus and density of the top solid. The constant y is a root of a complicated determinant, which is given by Eq. (5.110) in [22]. As is typical in wave propagation problems, the determinant is obtained from the interfacial boundary conditions and its roots correspond to nontrivial wave propagation speeds.

We have performed stability calculations where $\nu_{s,1}=\nu_{s,2}=0.01 \, \text{cm}^2/\text{s}$, $\nu_{p,1}=\nu_{p,2}=100 \, \text{cm}^2/\text{s}$, $\lambda_1=\lambda_2=1\times10^{-2}$ s, and $\sigma=10 \, \text{dyne/cm}$. The shear modulus, $\mu_j=\nu_{p,j}\rho_j/\lambda_j$, of each fluid is different, but the shear wave speeds, $(\mu_j/\rho_j)^{1/2}=(\nu_{p,j}/\lambda_j)^{1/2}$, are the same. The same range of forcing frequencies as before was considered and a linear relationship between the critical wave number and the vibration frequency was found. (The vibration frequency is the same as the standing wave frequency in this case since the waves respond harmonically to the forcing.) When $\rho_1=1 \, \text{g/cm}^3$ and $\rho_2=0.98 \, \text{g/cm}^3$, the slope of this relationship is approximately 103 cm/s. Using the moduli and densities of the two fluids, we have also computed the slope predicted by the Stoneley wave theory. This gives a slope of 100 cm/s, which is in good agreement with the stability calculations. Therefore, it appears possible to excite Stoneley like waves at the interface of two viscoelastic fluids undergoing vertical vibration. We have tried several other density ratios and also found good agreement between the two calculations.
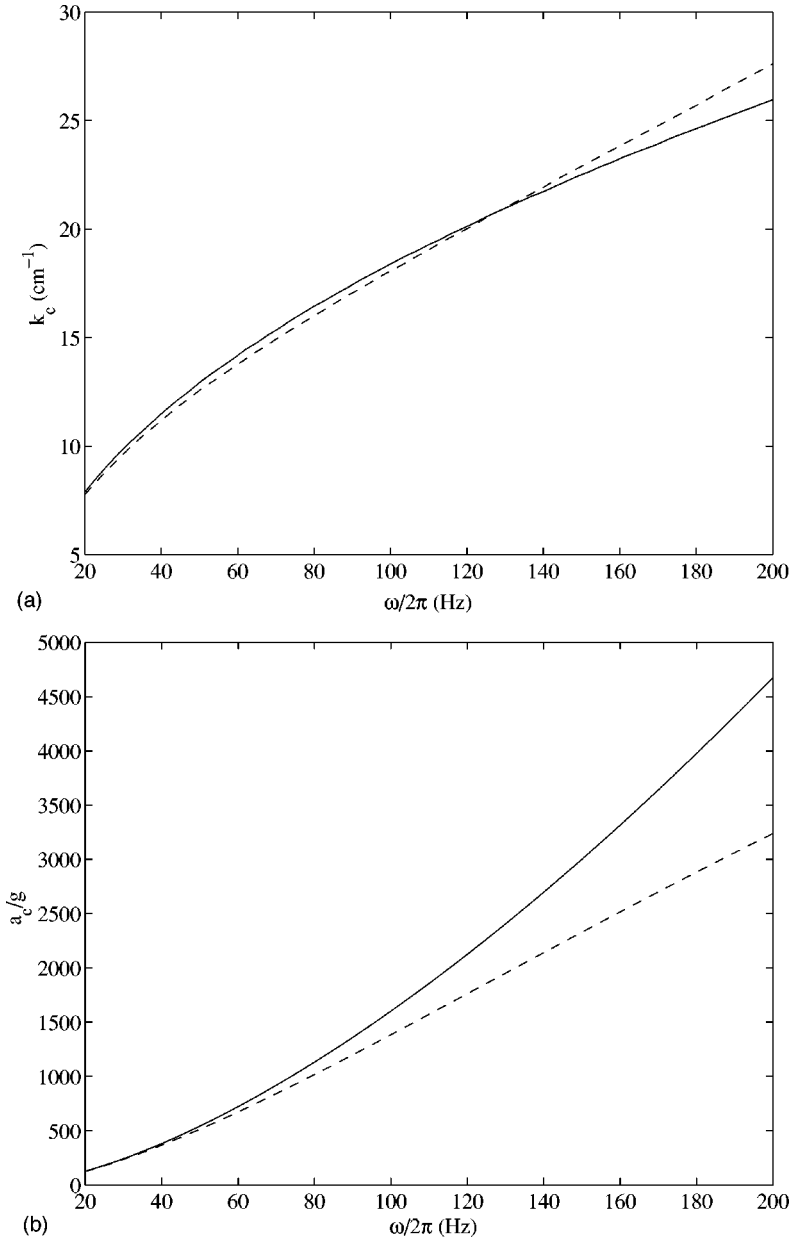
FIG. 4. Comparison of the (a) critical wave number (k_c) and (b) critical vibration amplitude (a_c) when the top fluid is viscoelastic (dashed line) and when it is Newtonian (solid line) but with the same zero-shear viscosity. The vibration frequency is ω.

IV. DISCUSSION AND CONCLUSIONS

When the interface between two viscoelastic fluids is subject to vertical vibration, it can become unstable and give rise to standing waves. Our linear stability analysis allows prediction of both the critical amplitude and the critical wave number, and this can be done for fluids of arbitrary depth and relaxation moduli. To elucidate the fundamental features of viscoelasticity, our results focused on infinite-depth fluids and made use of the single-mode Maxwell model.

When the bottom fluid is viscoelastic, the presence of a top Newtonian fluid makes it more difficult to excite the instability. It also shifts the critical wave numbers and the frequency of the subharmonic-to-harmonic transition. The behavior of the critical vibration amplitude with respect to the viscosity ratio and density ratio can be understood in terms of the corresponding Newtonian problem. The

shifts in the wave number and transition frequency could not have been anticipated from previous work.

If the bottom fluid is Newtonian and the top fluid is viscoelastic, it is possible for the surface waves to respond harmonically to the forcing if elastic forces are sufficiently strong relative to surface tension and viscous forces, and if the Deborah number of the top fluid is $O(1)$. We have also found that the harmonic response will not occur unless the top fluid is sufficiently dense compared to the bottom fluid, and that the critical amplitude, critical wave number, and transition frequency all depend on the density and viscosity ratios. If the level of elasticity is relatively weak, then the response will be subharmonic but the instability will be easier to excite compared to the case where one has a Newtonian top fluid of the same zero-shear viscosity. The mechanism responsible for this behavior is the same one that operates when only one fluid is present [9,10].

If both fluids are viscoelastic, Stoneley-like waves can be excited at the interface since the fluids can be made to behave like elastic solids. This is an extension of a previous discovery that Rayleigh-like waves can be excited when only one fluid is present [9].

The method developed in this paper should prove useful for understanding interfacial wave formation in applications where vertical vibration plays a role, including ultrasonic emulsification, microgravity science and engineering, and geophysics. For ultrasonic emulsification, one usually has $De_{1,2} \gg 1$. Thus, our results imply that the Newtonian theory should do an adequate job in predicting the initial instability if the solvent viscosity is used in the calculations. However, non-Newtonian effects will likely become important in the nonlinear regime of the instability as droplets pinch off from the interface. At mechanical frequencies, non-Newtonian effects should be more pronounced in the linear regime, and the present study provides a framework for understanding such effects.

[1] M. Faraday, Philos. Trans. R. Soc. London **121**, 319 (1831).

[2] T. B. Benjamin and F. Ursell, Proc. R. Soc. London, Ser. A **225**, 505 (1954).

[3] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, Oxford, 1960).

[4] J. Miles and D. Henderson, Annu. Rev. Fluid Mech. **22**, 143 (1990).

[5] K. Kumar, Proc. R. Soc. London, Ser. A **452**, 1113 (1996).

[6] J. Bechhoefer, V. Ego, S. Manneville, and B. Johnson, J. Fluid Mech. **288**, 325 (1995).

[7] M. Perlin and W. W. Schultz, Annu. Rev. Fluid Mech. **32**, 241 (2000).

[8] H. W. Muller and W. Zimmermann, Europhys. Lett. **45**, 169 (1999).

[9] S. Kumar, Phys. Fluids **11**, 1970 (1999).

[10] F. Raynal, S. Kumar, and S. Fauve, Eur. Phys. J. B **9**, 175 (1999).

[11] C. Wagner, H. W. Muller, and K. Knorr, Phys. Rev. Lett. **83**, 308 (1999).

[12] M. K. Li and H. S. Fogler, J. Fluid Mech. **88**, 499 (1978).

[13] G. V. Middleton and P. R. Wilcock, *Mechanics in the Earth and Environmental Sciences* (Cambridge University, Cambridge, 1994).

[14] F. Troyon and R. Gruber, Phys. Fluids **14**, 2069 (1971).

[15] F. M. Hoffmann and G. H. Wolf, J. Appl. Phys. **45**, 3859 (1974).

[16] D. Jacqmin and W. M. B. Duval, J. Fluid Mech. **196**, 495 (1988).

[17] E. Hasegawa, F. Nagashima, and H. Takashima, JSME Int. J. **30**, 1750 (1987).

[18] K. Kumar and L. S. Tuckerman, J. Fluid Mech. **279**, 49 (1994).

[19] S. Fauve, K. Kumar, C. Laroche, D. Beysens, and Y. Garrabos, Phys. Rev. Lett. **68**, 3160 (1992).

[20] R. B. Bird, R. C. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids, Volume 1: Fluid Mechanics* (Wiley, New York, 1987).

[21] R. Stoneley, Proc. R. Soc. London, Ser. A **106**, 416 (1924).

[22] J. D. Achenbach, *Wave Propagation in Elastic Solids* (North-Holland, Amsterdam, 1973).